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Optimal guidance law design for impact with terminal angle of attack constraint

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ABSTRACT

According to the requirements of the precision guidance of air-to-surface weapons with multiple constraints, a new optimal guidance law applying to attack the ground stationary target is designed. On the basis of dividing the 3D movement of the weapon into the movements of diving plane and turning plane, the relative motion of weapon-target is established first. Considering the conditions of miss distance, impact angle and terminal angle of attack, the general formulation of a new guidance law with an arbitrary system order is deduced by solving the Riccati equation of the quadratic optimal control. The approximate expressions of lag free system and first-order lag system are given. The validity of the optimal guidance law is verified by the comparable simulations of the characteristic trajectory. The simulation results shows that the optimal guidance laws satisfy the precision guidance with impact angle constraint as well as the angle of attack converges to zero at final time, which is important for warhead effect.

1. Introduction

The main objective of guidance systems for air-to-surface weapons is to generate suitable commands that produce zero terminal miss distance. In some cases, however, many kinds of guided weapons are expected not only to get a minimum miss distance but also to achieve a desired impact attitude angle, so that the warhead of the weapons can acquire a better kill effect. The impact precision requirements are so stringent that the conventional guidance laws, such as the proportional navigation guidance (PNG) law, cannot direct the vehicle to the target and achieve the desired impact direction.

Terminal guidance of reentry vehicles with constrained attitude angle at impact is studied by applying linear quadratic optimization techniques in Ref. [1], which seems to be a pioneering research on this area. For the past few decades, a variety of methods for guidance laws with terminal impact angle constraints have been extensively studied. Based on the different fundamental theories, the existing guidance laws are categorized into modified PNG laws, optimal guidance laws, variable structure guidance laws and other guidance laws.

The proportional navigation (PN) and its variants have been extensively studied and widely used as the homing guidance laws for short-range intercept because of their simple structure, mature theory and their ease of implementation. A variation of the conventional PNG law with supplementary time-varying bias is proposed in Ref. [2] to fulfill a special guidance goal that impact a target with a desired attitude angle. Based on the principle of following a circular arc to the target, a precision guidance law with impact angle constraint for a two-dimensional planar intercept is derived in Ref. [3]. Reference [4] considers the problem of guiding a hypersonic gliding vehicle in the terminal phase to a target location and derives an adaptive PN guidance to direct the vehicle to a ground target with specified impact direction. A track forecast guidance law is presented in Ref. [5] to meet the requirement of impact angle and to hit targets precisely based on traditional PNG.

The second category comprises the research in which the terminal impact angle controllers are given as optimal solutions by using optimal control theory and the associated linear quadratic optimal control theory [6–8]. A practical terminal guidance law is presented in Ref. [9] for impact angle control to enhance terminal effectiveness of anti-tank and anti-ship missile systems. An optimal guidance law with the capacity of adjusting terminal maneuver acceleration is studied in Ref. [10] to control impact angle as well as miss distance. A generalized formulation of energy minimization optimal guidance law for constant speed missiles with an

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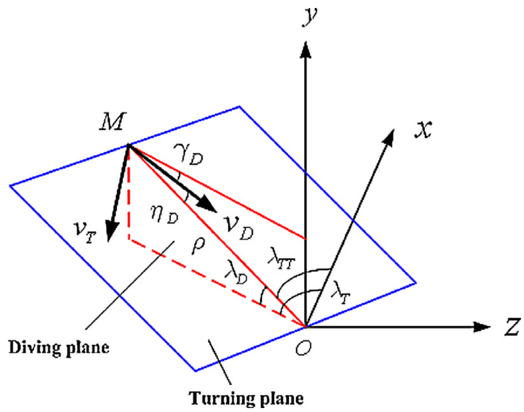

Fig. 1. Coordinate system and geometry.

arbitrary system order is proposed in Ref. [11] to achieve a desired impact angle as well as zero miss distance. An optimal guidance law with terminal constraints of miss distance and impact angle is obtained in Ref. [12] as the solution of a linear quadratic optimal control problem with the energy cost weighted by a power of the time-to-go. With the optimal control theory and precise linearization method, the reentry maneuverable warhead guidance scheme design problem is transformed to linear system synthesis problem in Ref. [13]. A three-dimensional guidance law with terminal impact angle constraint by using Lyapunov stability theory is proposed in Ref. [14]. Reference [15] deals with the guidance and control system to impact a target with a desired impact angle for precision guided bombs and the impact angle control guidance is derived by the solution to the linear quadratic optimal control problem.

The variable structure control (VSC) theory and associated sliding mode control (SMC) have been developed and applied enormously in the past few decades due to the inherent robustness and relatively simple control algorithm of the VSC system [16–18]. A 'homing' guidance with terminal angular constraint is stated in Ref. [19] by employing a Lyapunov-like function with the sliding mode control methodology. A variable structure midcourse guidance law with terminal angle constraint is proposed in Ref. [20] base on the concept of virtual target. Reference [21] presents a variable structure guidance law with constraints on impact angle which can be implemented conveniently with little target information. Reference [22] deals with a new passive homing guidance law for a stationary or a slowly moving target using sliding mode control technique and designs a guidance law that can control a terminal impact angle to maximize warhead effect. By combining the linear quadratic optimal theory and the VSC methodology, a 3-dimensional robust guidance law is developed for impact angle control in Ref. [23]. An approach to impact time and angle guidance through a combination of a novel line-of-sight (LOS) rate shaping technique and a new second-order sliding mode approach is presented in Ref. [24].

From above studies, we can notice that nearly all of them only take few constraints during the design of terminal guidance laws. But for the problem of the precision guidance with multiple constraints in the modern air-to-surface weapon, it is necessary for us to take into account multi-constraint conditions simultaneously, such as miss distance, impact angle, orientation angle, angle of attack, etc. References [11,25] deduce a new guidance law which considers the constraints of miss distance, impact angle and system dynamics, but they only gives an approximate expression of zero-order lag-free system and neglect the delay factors in missile system such as autopilots, actuators and dynamics.

In this paper, we try to add the terminal angle of attack constraint to the derivation of optimal guidance law in addition to the requirements of miss distance and impact angle. Section 2 describes the weapon-target engagement model and establishes the state equations of motion. Section 3 derives the general formulation of optimal guidance law with terminal angle of attack constraint by solving the Riccati equation of the linear quadratic optimal control. Considering the system dynamics and the lag time attributed to the transfer of input commands to output reaction, the analytical guidance formulations based on the lag free system assumption and first order lag system approximation are presented in Sections 4 and 5, respectively. The validity of the derived optimal guidance laws is verified by the comparable simulations of the characteristic trajectory with conventional optimal guidance laws in Section 6. The simulating results show that the angle of attack converges to zero at final time when applying the optimal guidance law with terminal angle of attack constraint, which satisfies the precision guidance and is important for warhead effect.

2. State equations of motion

In order to simplify the equations of the striking situation, the weapon-target engagement model is divided into diving plane and turning plane as shown in Fig. 1. An earth-fixed coordinate system is defined. The target is stationary and stays at the origin of the coordinate system. The y-axis is pointed to the up and deviates from the Earth center, the x-axis stays in the horizontal plane and is pointed to the weapon, and the z-axis completes the right-hand system. It is also assumed that the angle-of-attack is small and its velocity is constant.

The standard three-dimensional equations of motion of the guided weapon over a flat earth can be represented by the following nonlinear differential equations.

$$\begin{cases} \dot{v} = \dfrac{(P\cos\alpha\cos\beta - X)}{m} - \dfrac{\mu\left[x\cos\theta\cos\sigma + (y+R_0)\sin\theta - z\cos\theta\sin\sigma\right]}{r^3} \\ \dot{\theta} = \dfrac{(P\sin\alpha + Y)}{(mv)} + \dfrac{\mu\left[x\sin\theta\cos\sigma - (y+R_0)\cos\theta - z\sin\theta\sin\sigma\right]}{(r^3 v)} \\ \dot{\sigma} = \dfrac{P\cos\alpha\sin\beta - Z}{mv\cos\theta} - \dfrac{\mu}{r^3 v\cos\theta}(x\sin\sigma + z\cos\sigma) \\ \dot{x} = v\cos\theta\cos\sigma \quad \dot{y} = v\sin\theta \quad \dot{z} = -v\cos\theta\sin\sigma \end{cases} \tag{1}$$

where $\dot{x}$, $\dot{y}$, $\dot{z}$ are the change rate of position coordinates x, y, z, respectively. v is the earth relative velocity and $\dot{v}$ denotes the acceleration. The flight path angle and angular rate are θ and $\dot{\theta}$, while the velocity deflection angle and angular rate are σ and $\dot{\sigma}$. α and β are the angle-of-attract and angle-of-sideslip respectively. X, Y and Z represent the drag, lift and side force. r is the radial distance from the center of the earth to the weapon and μ is the gravitational constant while R_0 denotes the mean radius of the Earth. The propulsion P and mass m are all constants.

As shown in Fig. 1, the motion of weapon can be decomposed in diving plane and turning plane. The LOS ρ from the target to the weapon is defined by elevation angle λ_D and azimuth angle λ_T. v_D and v_T are the velocity decomposition respectively in diving plane and turning plane. λ_{TT} is the azimuth angle in the turning plane. The equation of weapon-target relative movement can be found in Ref. [26].

$$\begin{cases} \ddot{\lambda}_D = \left(\dfrac{\dot{v}}{v} - 2\dfrac{\dot{\rho}}{\rho}\right)\dot{\lambda}_D - \dot{\rho}\dfrac{\dot{\gamma}_D}{\rho} \\ \ddot{\lambda}_{TT} = \left(\dfrac{\dot{v}}{v} - 2\dfrac{\dot{\rho}}{\rho}\right)\dot{\lambda}_{TT} + \dot{\rho}\dfrac{\dot{\gamma}_T}{\rho} \end{cases} \tag{2}$$

where γ_D and γ_T denote the azimuth angle of velocity in diving plane and turning plane respectively, and their angular rate are $\dot{\gamma}_D$ and $\dot{\gamma}_T$. $\dot{\rho}$ is the change rate of LOS. $\dot{\lambda}_D$ and $\ddot{\lambda}_D$ are the angular rate and angular acceleration of elevation angle λ_D, $\dot{\lambda}_{TT}$ and $\ddot{\lambda}_{TT}$ denote the angular rate and angular acceleration of azimuth angle in the turning plane λ_{TT}.

It is known that the movement of ground striking weapon is mainly in the diving plane and the side guidance has less impact on the precision of miss distance and terminal attitude angle, so we only discuss the optimal guidance law derivation of diving plane. From (2) and the velocity is assumed to be a constant, the state equations of motion in diving plane are as follows

$$\dot{x}_D = \boldsymbol{A}_D\boldsymbol{x}_D + \boldsymbol{B}_D\boldsymbol{u}_D \tag{3}$$

where $\boldsymbol{x}_D = \begin{pmatrix} \lambda_D + \gamma_{DF} \\ \dot{\lambda}_D \end{pmatrix}$, $\boldsymbol{A}_D = \begin{pmatrix} 0 & 1 \\ 0 & 2/T_g \end{pmatrix}$, $\boldsymbol{B}_D = \begin{pmatrix} 0 \\ 1/T_g \end{pmatrix}$, $\boldsymbol{u}_D = \dot{\gamma}_D$, $T_g = -\dfrac{\rho}{\dot{\rho}}$.$\gamma_{DF}$ denotes the desired impact angle, which is a constant since the target does not maneuver.

In addition to the miss distance and impact angle constraints, zero angle of attack constraint should also be considered in the design of terminal guidance law. In order to take these three constraints into account, following performance index should be selected.

$$J = \frac{\left[X(t_f) - X_f\right]^T F\left[X(t_f) - X_f\right]}{2} + \frac{\int_0^{t_f} \boldsymbol{u}^T(\tau) L u(\tau) d\tau}{2} \tag{4}$$

where $X = (\lambda_D + \gamma_{DF}, \dot{\lambda}_D, \alpha)^T$. F and L denote the symmetric nonnegative and positive definite matrices, respectively. t_f and X_f are the time of flight and the specified terminal constraints, respectively. To ensure $X(t_f) \to (0, 0, 0)^T$, the optimal control should be derived. Because the angle of attack cannot be measured and controlled easily, we cannot choose angle of attack as the direct control variable. Based on the aerodynamics, the following approximate expression holds.

$$Y = qS(C_y^\alpha \alpha + C_y^\delta \delta)$$

where q is the dynamic pressure and S denotes the reference area. C_y^α and C_y^δ are the aerodynamic coefficients. δ is control piston deflexion angle. Under the balance state, $C_y^\delta \delta$ is relative small, so the terminal zero angle of attack condition can be transformed to the problem of nulling terminal acceleration command.

In order to achieve the goal of angle of attack control, the dynamic characteristic of the weapon should be introduced into the movement model. The dynamics of the weapon system with scalar input $u_D(t)$ can be represented by Ref. [27].

$$\begin{pmatrix} \dot{x}_{D1} \\ \dot{x}_{D2} \end{pmatrix} = \begin{pmatrix} \dot{a}_D \\ \dot{p}_D \end{pmatrix} = \begin{pmatrix} a_{11} & \boldsymbol{a}_{12} \\ \boldsymbol{a}_{21} & \boldsymbol{a}_{22} \end{pmatrix} \begin{pmatrix} a_D \\ \boldsymbol{p}_D \end{pmatrix} + \begin{pmatrix} b_1 \\ \boldsymbol{b}_2 \end{pmatrix} u_D(t) \tag{5}$$

where the acceleration a_D is the first state variable and the vector $\boldsymbol{P}_D$ consists of the rest n-1 state variables. Therefore, a_{11} and b_1 are scalars; $\boldsymbol{a}_{12}$, $\boldsymbol{a}_{21}^T$ and $\boldsymbol{b}_2$ are $(n-1)\times 1$ vectors; $\boldsymbol{a}_{22}$ is a $(n-1)\times(n-1)$ matrix.

Because $a_D \approx -\dot{\rho}\dot{\gamma}_D$ and $\dot{\gamma}_D$ is proportion to α, then combine (3) with (5), we can get the general state equations of motion in diving plane.

$$\dot{X} = \boldsymbol{AX} + \boldsymbol{Bu} \tag{6}$$

where $\boldsymbol{X}=(x_1,x_2,x_3,\boldsymbol{x}_4)^T=(\lambda_D+\gamma_{DF},\dot{\lambda}_D,a_D,\boldsymbol{p}_D)^T$, $\boldsymbol{A}=\begin{bmatrix}0&1&0&0\\0&2/T_g&1/\rho&0\\0&0&a_{11}&a_{12}\\0&0&a_{21}&a_{22}\end{bmatrix}$, $\boldsymbol{B}=(0\quad 0\quad b_1\quad b_2)^T$, $u=a_D$.

Considering the multi-constraint conditions of $\lambda_D+\gamma_{DF}=0$, $\lambda_D=0$ and $\alpha=0$ $(a_D=0)$ at impact moment, the terminal constraints of (5) is

$$X_f=(0,0,0,x_{4f})^T \tag{7}$$

3. General formulation of optimal guidance laws

Based on the linear differential equation in the air-to-surface striking problem given above, let us consider the following optimal control problem with terminal acceleration constraint: in order to achieve better kill effect, we choose (4) as the performance index, which reflects the minimum miss distance and energy control. Then our goal is to find the optimum control law which minimizes (4) subject to (6).

This is a linear quadratic optimal control problem and its solution is a two-point boundary value problem, which can be expressed in the form of a Riccati equation [28,29]. By using the analytic representation of its solution [30], it is easy to show that [27]

$$u^*=-\boldsymbol{L}^{-1}\boldsymbol{B}^T\Phi^T(t_f,t)\boldsymbol{F}\left[\boldsymbol{X}(t_f)-\boldsymbol{X}_f\right] \tag{8}$$

where

$$\begin{aligned}&X(t_f)=\left[\boldsymbol{I}+\int_t^{t_f}\Phi(t_f,\tau)\boldsymbol{B}\boldsymbol{L}^{-1}\boldsymbol{B}^T\Phi^T(t_f,\tau)\boldsymbol{F}d\tau\right]^{-1}\times\left[\Phi(t_f,t)\boldsymbol{X}(t)-\boldsymbol{X}_f\right]+\boldsymbol{X}_f\\&\dot{\Phi}(t,t_0)=\boldsymbol{A}\Phi(t,t_0),\quad \Phi(t,t_0)=I\end{aligned} \tag{9}$$

The term $\Phi(t_f,t)\,\boldsymbol{X}(t)$ is the predicted state without control effect, i.e. $\mathbf{u}(\tau)=0(t\le\tau\le t_f)$. The quantity pre-multiplying it in (8) can be viewed as a time varying gain matrix.

Because we are concerned about the minimization of the miss distance, the terminal impact angle and terminal acceleration, the corresponding terminal constraints become $X_f=(0,0,0,x_{4f})^T$, thus the weighting matrices $\boldsymbol{F}$ and $\boldsymbol{L}$ can be chosen as

$$F=\begin{bmatrix}f_1&0&0&0\\0&f_2&0&0\\0&0&f_3&0\\0&0&0&0\end{bmatrix},\quad L=1. \tag{10}$$

From (6), we can get transfer functions as shown by (11).

$$\begin{cases}\dfrac{x_1(s)}{x_2(s)}=\dfrac{1}{s}\\[2mm]\dfrac{x_2(s)}{x_3(s)}=\dfrac{1}{\rho(s-2/T_g)}\\[2mm]\dfrac{x_1(s)}{x_3(s)}=\dfrac{1}{\rho s(s-2/T_g)}\end{cases}. \tag{11}$$

Let us $\boldsymbol{X}(s)$ and $u(s)$ be Laplace transform of state vector $\boldsymbol{X}(s)$ and scalar input $u(s)$, respectively, then for zero initial condition, the state transition matrix can be provided by inverse Laplace transform.

$$\Phi(t_f,t)=\mathfrak{L}^{-1}\left\{sI-A\right\}^{-1}_{t_f,t}=\mathfrak{L}^{-1}\left\{\frac{X(s)}{X(0)}\right\}_{t_f,t}=\begin{bmatrix}1&t_f-t&\phi_{13}&\varphi_{14}\\0&1&\phi_{23}&\varphi_{24}\\0&0&\phi_{33}&\varphi_{34}\\0&0&\phi_{43}&\varphi_{44}\end{bmatrix} \tag{12}$$

$$\Phi(t_f,t)B=\mathfrak{L}^{-1}\left\{sI-A\right\}^{-1}_{t_f-t}B=\mathfrak{L}^{-1}\left\{\frac{X(s)}{u(s)}\right\}_{t_f-t}=\mathfrak{L}^{-1}\left\{\Xi_1(s)\frac{a_D(s)}{u(s)};\Xi_2(s)\frac{a_D(s)}{u(s)};\frac{a_D(s)}{u(s)};\frac{p_D(s)}{u(s)}\right\}_{t_f-t} \tag{13}$$

where

$$\phi_{13}=L^{-1}\left\{\Xi_1(s)\frac{a_D(s)}{a_D(0)}\right\}_{t_f-t},\quad \phi_{23}=L^{-1}\left\{\Xi_2(s)\frac{a_D(s)}{a_D(0)}\right\}_{t_f-t},\quad \phi_{33}=L^{-1}\left\{\frac{a_D(s)}{a_D(0)}\right\}_{t_f-t},\quad \phi_{43}=L^{-1}\left\{\frac{p_D(s)}{a_D(0)}\right\}_{t_f-t}$$

$$\varphi_{14}=L^{-1}\left\{\Xi_1(s)\frac{a_D(s)}{p_D(0)}\right\}_{t_f-t},\quad \varphi_{24}=L^{-1}\left\{\Xi_2(s)\frac{a_D(s)}{p_D(0)}\right\}_{t_f-t},\quad \varphi_{34}=L^{-1}\left\{\frac{a_D(s)}{p_D(0)}\right\}_{t_f-t},\quad \varphi_{44}=L^{-1}\left\{\frac{p_D(s)}{p_D(0)}\right\}_{t_f-t}$$

$\Xi_1(s)=x_1(s)/x_3(s)$ is the transfer function from a_D to $(\lambda_D+\gamma_{DF})$, $\Xi_2(s)=x_2(s)/x_3(s)$ the transfer function from a_D to λ_D, $a_D(s)/a_D(0)$ the autopilot acceleration response to initial condition in the acceleration state, and $a_D(s)/\boldsymbol{p}_D(0)$ the autopilot acceleration response to initial conditions in the states $\boldsymbol{p}_D$, and $\boldsymbol{p}_D(s)/a_D(0)$ the states response to initial conditions in the acceleration state, $\boldsymbol{p}_D(s)/\boldsymbol{p}_D(0)$ the states response to initial conditions in states $\boldsymbol{p}_D$, $a_D(s)/u(0)$ is autopilot transfer function.

Substituting (7), (10) and (13) into (8), a general expression of the optimal control is given by

$$u^* = -\boldsymbol{L}^{-1}\boldsymbol{B}^T\Phi^T(t_f,t)\boldsymbol{F}\left[\boldsymbol{X}(t_f)-\boldsymbol{X}_f\right] = -\left[f_1x_1(t_f)\mathrm{L}^{-1}\left\{\Xi_1(s)\frac{a_D(s)}{u(s)}\right\}_{t_f-t}+f_2x_2(t_f)\mathrm{L}^{-1}\left\{\Xi_2(s)\frac{a_D(s)}{u(s)}\right\}_{t_f-t}+f_3x_3(t_f)\mathrm{L}^{-1}\left\{\frac{a_D(s)}{u(s)}\right\}_{t_f-t}\right] \tag{14}$$

In order to avoid the inverse operation of 4×4 matrix, we should multiply the two sides of (9) by $I+\int_t^{t_f}\Phi(t_f,\tau)\boldsymbol{B}\boldsymbol{L}^{-1}\boldsymbol{B}^T\Phi^T(t_f,\tau)\boldsymbol{F}d\tau$, then the terminal states $x_1(t_f)$, $x_2(t_f)$ and $x_3(t_f)$ can be obtained by substituting (7), (10), (12) and (13) into (9)

$$x_1(t_f)=\frac{1}{\Delta}\left[C_{11}x_1+\left[C_{11}(t_f-t)+C_{12}\right]x_2+(C_{11}\phi_{13}+C_{12}\phi_{23}+C_{13}\phi_{33})x_3+(C_{11}\varphi_{14}+C_{12}\varphi_{24}+C_{13}\varphi_{34})x_4\right] \tag{15}$$

$$x_2(t_f)=\frac{1}{\Delta}\left[C_{21}x_1+\left[C_{21}(t_f-t)+C_{22}\right]x_2+(C_{21}\phi_{13}+C_{22}\phi_{23}+C_{23}\phi_{33})x_3+(C_{21}\varphi_{14}+C_{22}\varphi_{24}+C_{23}\varphi_{34})x_4\right] \tag{16}$$

$$x_3(t_f)=\frac{1}{\Delta}\left[C_{31}x_1+\left[C_{31}(t_f-t)+C_{32}\right]x_2+(C_{31}\phi_{13}+C_{32}\phi_{23}+C_{33}\phi_{33})x_3+(C_{31}\varphi_{14}+C_{32}\varphi_{24}+C_{33}\varphi_{34})x_4\right] \tag{17}$$

where

$$\begin{aligned}
&\Delta = K_{11}K_{22}K_{33}+K_{12}K_{23}K_{31}+K_{13}K_{21}K_{32}-K_{13}K_{22}K_{31}-K_{12}K_{21}K_{33}-K_{11}K_{23}K_{32}\\
&C_{11}=K_{22}K_{33}-K_{23}K_{32},\ C_{12}=K_{13}K_{32}-K_{12}K_{33},\ C_{13}=K_{12}K_{23}-K_{13}K_{22}\\
&C_{21}=K_{23}K_{31}-K_{21}K_{33},\ C_{22}=K_{11}K_{33}-K_{13}K_{31},\ C_{23}=K_{13}K_{21}-K_{11}K_{23}\\
&C_{31}=K_{21}K_{32}-K_{22}K_{31},\ C_{32}=K_{12}K_{31}-K_{11}K_{32},\ C_{33}=K_{11}K_{22}-K_{12}K_{21}\\
&K_{11}=1+f_1\int_t^{t_f}L_1^2d\tau,\ K_{12}=f_2\int_t^{t_f}L_1L_2d\tau,\ K_{13}=f_3\int_t^{t_f}L_1L_3d\tau\\
&K_{21}=f_1\int_t^{t_f}L_1L_2d\tau,\ K_{22}=1+f_2\int_t^{t_f}L_2^2d\tau,\ K_{23}=f_3\int_t^{t_f}L_2L_3d\tau\\
&K_{31}=f_1\int_t^{t_f}L_1L_3d\tau,\ K_{32}=f_2\int_t^{t_f}L_2L_3d\tau,\ K_{33}=1+f_3\int_t^{t_f}L_3^2d\tau\\
&L_1=L^{-1}\left\{\Xi_1(s)\frac{a_D(s)}{u(s)}\right\}_{t_f-t},\quad L_2=L^{-1}\left\{\Xi_2(s)\frac{a_D(s)}{u(s)}\right\}_{t_f-t},\quad L_3=L^{-1}\left\{\frac{a_D(s)}{u(s)}\right\}_{t_f-t}
\end{aligned}$$

Eqs. (14–17) are the general formulation of terminal optimal guidance law with arbitrary system order to minimize the terminal acceleration in addition to the miss distance and terminal impact angle of the weapon. We can easily prove that the terminal states $x_1(t_f)$, $x_2(t_f)$ and $x_3(t_f)$ remain constant when the optimal control u^* is applied, so the optimal control u^* is a linear combination of a step response, a ramp response and in impulse response of the weapon. Obviously, if we set f_2=f_3=0, the resultant optimal guidance law only concerns minimizing miss distance and is approximately equals to PNG [27]. And if we have only f_3=0, the resultant optimal guidance law concerns minimizing miss distance and terminal impact angle simultaneously, and is approximately equals to the guidance law of [11].

4. Optimal guidance law for lag free system

Because there are many delay factors in guided weapon system such as autopilot, actuator and dynamics, the guidance law for lag free system may seems to weaken its real applications. In this section, however, we first derive the optimal guidance law for lag free system because of its simplicity, and then we investigate the specific formulation of the optimal guidance law for first-order lag system in the next section.

If the transfer function of the weapon is assumed as lag free system

$$\frac{a_D(s)}{u(s)}=1 \tag{18}$$

Then, we can neglect x_3, x_4 and f_3. Substituting (18) into (15) and (16), and considering T_g=t_f-t and let $f_1\to\infty$, $f_2\to\infty$, we can obtain

$$\begin{aligned}
G_1&=\frac{2\rho^2}{T_g^3}(1+e^2)x_1(t)+\frac{\rho^2}{T_g^2}(3+e^2)x_2(t)\\
G_2&=\frac{\rho^2}{T_g^2}(1+e^2)x_1(t)-\frac{\rho^2}{2T_g}\frac{5-e^4}{1-e^2}x_2(t)
\end{aligned}$$

Substituting last two equations into (14–17), the optimal guidance control for lag free system can be calculated such that

$$a_D^*=u^*=(1-e^2)\frac{\rho}{T_g^2}x_1(t)+\frac{3+e^4}{2(1-e^2)}\frac{\rho}{T_g}x_2(t) \tag{19}$$

where e is the base of index. Moreover, considering (19) and the approximate relation $a_D \approx -\dot{\rho}\dot{\gamma}_D$, the simple form of the optimal guidance law can be obtained as follows.

$$\dot{\gamma}_D = -\frac{a_D^*}{\dot{\rho}} = (1-e^2)\frac{\lambda_D+\gamma_{DF}}{T_g} + \frac{3+e^4}{2(1-e^2)}\dot{\lambda}_D \approx -6.389\frac{\lambda_D+\gamma_{DF}}{T_g} - 4.508\dot{\lambda}_D \tag{20}$$

Eq. (20) is the optimal guidance law in diving plane for lag free system. It is equivalent to an amendatory proportional guidance law with a terminal angular constraint item, which not only satisfies the miss distance and impact angle constraints, but also minimizes the control energy cost.

5. Optimal guidance law for first-order lag system

Although the guidance laws for lag free system are still used in real application because of its simplicity, its simple readability comes at the expense of performance. Furthermore, the development of modern guidance laws comes from the requirement for better performance, which can be achieved by consideration of the detailed dynamics of the weapon. Considering a higher order lag system when developing the optimal guidance law can improve the stability and the command behavior near the impact instant, so it is necessary for us to derive the formulation of the optimal guidance law for higher order lag system and investigate its properties.

We cannot exactly model the system delay, but the weapon transfer function can be approximated by first-order lag system as follows.

$$\frac{a_D(s)}{u(s)} = \frac{1}{Ts+1} \tag{21}$$

where T is the time constant. Then we can eliminate $\boldsymbol{x}_4$. The optimal guidance command under this approximation can be derived by substituting (21) into (14–17) as shown by (22). Here we consider $T_g = t_f - t$ and let $f_1 \to \infty$, $f_2 \to \infty$, $f_3 \to \infty$.

$$a_D^* = -\frac{1}{\Delta}\left[G_1(t)x_1(t) + G_2(t)x_2(t) + G_3(t)x_3(t)\right] = -\frac{1}{\Delta}\left[G_1(t)(\lambda_D+\gamma_{DF}) + G_2(t)\dot{\lambda}_D + G_3(t)a_D\right] \tag{22}$$

where

$$\Delta = A_{11}A_{22}A_{33} + 2A_{12}A_{13}A_{23} - A_{13}^2A_{22} - A_{12}^2A_{33} - A_{23}^2A_{11}$$

$$G_1(t) = L_1G_{11} + L_2G_{12} + L_3G_{13},\ G_2(t) = L_1G_{21} + L_2G_{22} + L_3G_{23},\ G_3(t) = L_1G_{31} + L_2G_{32} + L_3G_{33}$$

$$G_{11} = A_{22}A_{33} - A_{23}^2,\ G_{21} = (A_{22}A_{33} - A_{23}^2)T_g + (A_{13}A_{23} - A_{12}A_{33})$$

$$G_{31} = \phi_{13}(A_{22}A_{33} - A_{23}^2) + \phi_{23}(A_{13}A_{23} - A_{12}A_{33}) + \phi_{33}(A_{12}A_{23} - A_{13}A_{22})$$

$$G_{12} = A_{13}A_{23} - A_{12}A_{33},\ G_{22} = (A_{13}A_{23} - A_{12}A_{33})T_g + (A_{11}A_{33} - A_{13}^2)$$

$$G_{32} = \phi_{13}(A_{13}A_{23} - A_{12}A_{33}) + \phi_{23}(A_{11}A_{33} - A_{13}^2) + \phi_{33}(A_{12}A_{13} - A_{11}A_{23})$$

$$G_{13} = A_{12}A_{23} - A_{13}A_{22},\ G_{23} = (A_{12}A_{23} - A_{13}A_{22})T_g + (A_{12}A_{13} - A_{11}A_{23})$$

$$G_{33} = \phi_{13}(A_{12}A_{23} - A_{13}A_{22}) + \phi_{23}(A_{12}A_{13} - A_{11}A_{23}) + \phi_{33}(A_{11}A_{23} - A_{12}^2)$$

If we let $a = -2/T_g$, $b = 1/T$, then

$$A_{11} = \frac{1}{\rho^2}\left[\frac{1}{a^2}T_g + \frac{b^2(1-e^4)}{2a^3(a-b)^2} + \frac{2b(1-e^2)}{a^3(a-b)} + \frac{1-e^{-2bT_g}}{2b(a-b)^2} - \frac{1-e^{-bT_g}}{ab(a-b)} - \frac{2b(1-e^{2-bT_g})}{a(a-b)^2(a+b)}\right]$$

$$A_{12} = \frac{b}{\rho^2(b-a)}\left[\frac{b(1-e^4)}{2a^2(a-b)} + \frac{1-e^2}{a^2} + \frac{1-e^{-2bT_g}}{2b(a-b)} - \frac{1-e^{-bT_g}}{ab} - \frac{1-e^{2-bT_g}}{a(a-b)}\right]$$

$$A_{13} = \frac{b}{\rho}\left[\frac{1-e^{-bT_g}}{ab} + \frac{b(1-e^{-2bT_g})}{a(a-b)(a+b)} - \frac{1-e^{2-bT_g}}{2b(a-b)}\right],\quad A_{22} = \frac{b^2}{\rho^2(b-a)^2}\left[\frac{1-e^4}{2a} + \frac{1-e^{-2bT_g}}{2b} - \frac{2(1-e^{2-bT_g})}{a+b}\right]$$

$$A_{23} = \frac{b^2}{\rho(b-a)}\left[\frac{1-e^{2-bT_g}}{a+b} - \frac{1-e^{-2bT_g}}{2b}\right],\quad A_{33} = \frac{b}{2}(1-e^{-2bT_g})$$

$$L_1 = \frac{1}{\rho}\left[\frac{1}{a} + \frac{b}{a(a-b)}e^2 - \frac{1}{(a-b)}e^{-bT_g}\right],\quad L_2 = \frac{b}{\rho(b-a)}(e^2 - e^{-bT_g}),\quad L_3 = be^{-bT_g}$$

Let $a_D(s)/a_D(0) = 1/s$, then

$$\phi_{13} = \frac{1}{\rho a^2}(e^3 - 3),\quad \phi_{23} = \frac{1}{\rho a}(1-e^2),\quad \phi_{33} = 1$$

Also taking into account the approximate relation $a_D \approx -\dot{\rho}\dot{\gamma}_D$, we can obtain the simple form of the optimal guidance law.

$$\dot{\gamma}_D = \frac{1}{\dot{\rho}\Delta}\left[G_1(t)(\lambda_D+\gamma_{DF}) + G_2(t)\dot{\lambda}_D + G_3(t)a_D\right] \tag{23}$$

Eq. (23) represents the optimal guidance law with miss distance, impact angle and terminal acceleration constraints for first-order lag system, which is equivalent to an amendatory proportional guidance law with two terminal constraint items and has the feedback form of state variables such as position, velocity and acceleration.

Table 1
Numerical values for simulation.

Symbol	Variable	Value (unit)
x_0, y_0, z_0	Weapon position	(10,16,−10) (km)
v_0	Weapon velocity	1360 (m/s)
θ_0	Initial heading angle	−10 (deg)
σ_0	Initial deflection angle	−120 (deg)
γ_{DF}	Impact angle	−88 (deg)

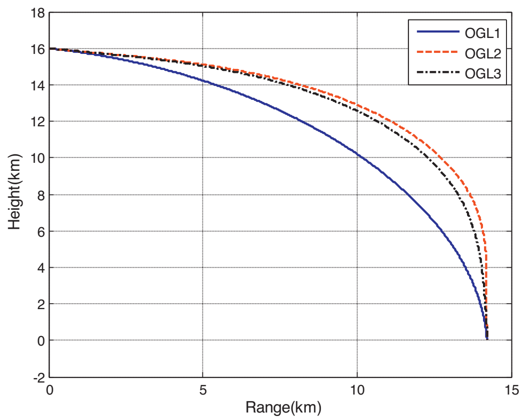

Fig. 2. Comparison of height-range profiles for the ground striking.

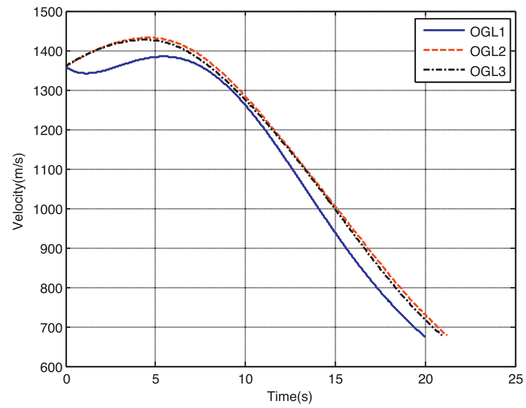

Fig. 3. Comparison of velocity profiles for the ground striking.

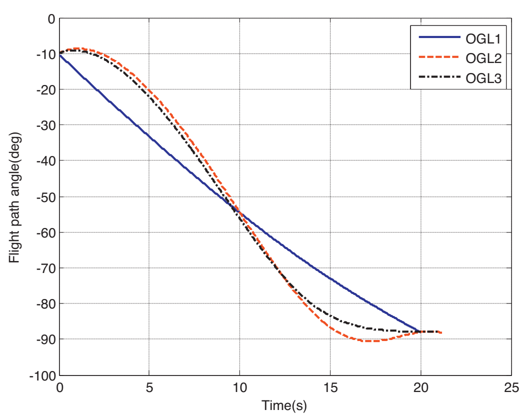

Fig. 4. Comparison of flight path angle profiles for the ground striking.

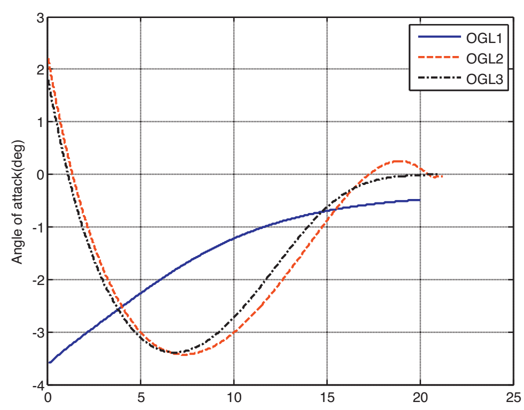

Fig. 5. Comparison of angle of attack profiles for the ground striking.

6. Simulation results

In order to investigate the properties of the derived optimal guidance laws with terminal angle of attack constraint, we compare them with the optimal guidance law derived in Ref. [26] through 3-dimensional nonlinear simulations with the initial conditions shown in Table 1. We choose the conventional navigation guidance law as the common guidance of these three cases in the turning plane so as to enhance the comparability.

In these simulations, three different kinds of optimal guidance laws to control impact angle and terminal angle of attack under the consideration of the initial conditions above are employed for performance comparison: OGL1 stands for the optimal guidance law presented in Ref. [26], OGL2 for the optimal guidance law for lag free system shown by (20), and OGL3 for the optimal guidance law for first-order lag system shown by (23). It is assumed that the weapon is a first-order lag system with 0.5 s time constant and all the state variables can be perfectly measured. These simulation tasks are carried on a PC with 2G memories and Intel(R) Core(TM) 2 Duo CPU. The version of MATLAB is R2009a.

The simulation results are presented by Figs. 2–6 and Table 2.

Fig. 2 represents the trajectories of the missile under the above conditions using OGL1, OGL2 and OGL3. All of the guidance laws complete guidance goal together. It is observed that the trajectories of the weapon using OGL2 and OGL3 deviate from initial LOS more than the other one. Fig. 3 shows the velocity profiles of the weapon. In the cases of the two derived guidance laws OGL2 and OGL3, The average magnitude of velocity is bigger than OGL1. Fig. 4 shows flight path angle profiles of these three guidance laws. All of them meet the terminal impact angle requirement, but the flight path angle variation curve of OGL1 is more stable than others. Figs. 5 and 6 represent the angle of attack profiles and the acceleration profiles, respectively. In the cases of OGL2 and OGL3, their maximum magnitude of acceleration in terminal

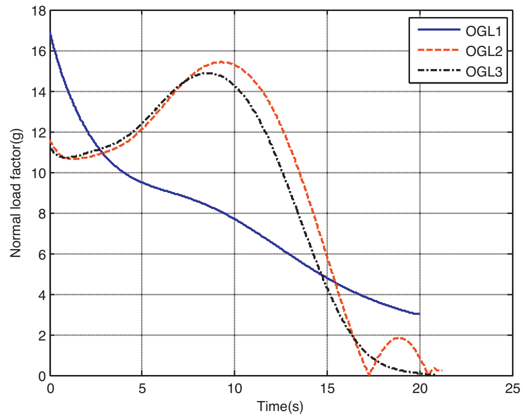

Fig. 6. Comparison of normal load factor profiles for the ground striking.

Table 2
Comparison of simulation results.

OGL	v_f(m/s)	$\triangle\theta_f$(deg)	CEP (m)	α_f(deg)
1	674.1257	−0.0169	0.0791	−0.4902
2	677.0112	−0.0609	0.0535	−0.0401
3	677.8680	−0.0052	0.0693	−0.0042

phase is smaller than OGL1 and converges to zero at final time as well. This is the most distinct property of the proposed optimal guidance laws with terminal angle of attack.

By analyzing the simulation results, we can find that the two derived optimal guidance laws with terminal angle of attack constraint have equivalent hit accuracy, impact angle error and terminal velocity when compared with the conventional guidance law with terminal impact angle constraint, but possess better performance on the terminal angle of attack control. These new derived guidance laws can achieve zero angle of attack impact on the whole, which play an important role in the air-to-surface penetrating weapons. Moreover, when comparing the simulation results of OGL2 and OGL3, the OGL3 for first-order lag system have higher stability and better guidance performance at final time, which proves that considering a higher order lag system when deriving the optimal guidance law can improve the stability and the command behavior near the impact instant.

7. Conclusion

In this paper, a general formulation of terminal optimal guidance law with an arbitrary system order is deduced by solving the Riccati equation of the linear quadratic optimal control problem to null the miss distance and achieve a desired impact angle as well as control the terminal angle of attack. The approximate expressions of the optimal guidance laws for lag free system and first-order lag system are given. The performance of the derived optimal guidance laws with terminal angle of attack constraint are investigated by comparable simulations with the conventional terminal guidance law. An important feature of the derived optimal guidance laws with terminal angle of attack is a linear combination of a step response, a ramp response and in impulse response of the weapon. The most distinct property of the proposed optimal guidance laws with terminal angle of attack is the magnitude of angle of attack in terminal phase converges to zero at final time.

Furthermore, we can find from the simulation results that the proposed energy minimization optimal guidance law for first-order lag system can not only shows small CEP and angle of attack at final time, but also presents good guidance performance, which proves that guidance laws for higher order lag system have good command behavior and robustness to system disturbances and uncertainties in terminal phase. However, the guided weapons must face up to some severe conditions and typical constraints such as heating rate constraint, normal load factor constraint and dynamic pressure constraint that have to be considered during the on-line applications. So it is necessary for us to develop appropriate guidance laws which can not only satisfy multiple constraints but also minimize specific performance index.

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